CHAPTER

13 Three Dimensional Geometry

1. **Vector Representation of a Point :** Position vector of point $P(x, y, z)$ is $x\hat{i} + y\hat{j} + z\hat{k}$.

2. **Distance Formula**

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, AB = \left|\overrightarrow{OB} - \overrightarrow{OA}\right|$$

3. **Distance of P from Coordinate Axes**

$$PA = \sqrt{y^2 + z^2}, PB = \sqrt{z^2 + x^2}, PC = \sqrt{x^2 + y^2}$$

4. **Section Formula :** $x = \dfrac{mx_2 + nx_1}{m+n}, y = \dfrac{my_2 + ny_1}{m+n}, z = \dfrac{mz_2 + nz_1}{m+n}$

   Mid Point : $x = \dfrac{x_1 + x_2}{2}, y = \dfrac{y_1 + y_2}{2}, z = \dfrac{z_1 + z_2}{2}$

5. **Direction Cosines and Direction Ratios**

   (*i*) **Direction cosines :** Let α, β, γ be angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then cos α, cosβ, cos γ are called the direction cosines of the line. The direction cosines are usually denoted by (ℓ, m, n). Thus $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

   (*ii*) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$.

   (*iii*) **Direction ratios:** Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

   (*iv*) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\ell = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

   (*v*) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \dfrac{x_2 - x_1}{|PQ|}$, $m = \dfrac{y_2 - y_1}{|PQ|}$ and $n = \dfrac{z_2 - z_1}{|PQ|}$.

6. **Angle between Two Line Segments**

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|.$$

   The line will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$, parallel if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

7. **Projection of a Line Segment on a Line :** If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then the projection of PQ on a line having direction cosines ℓ, m, n is $\left|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)\right|$.

8. **Equation of a Plane :** General form : $ax + by + cz + d = 0$, where a, b, c are not all zero, $a, b, c, d \in R$.

   (*i*) Normal form : $\ell x + my + nz = p$

   (*ii*) Plane through the point (x_1, y_1, z_1) :
   $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

   (*iii*) Intercept Form: $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

   (*iv*) Vector form: $(\vec{r} - \vec{a})\cdot\vec{n} = 0$ or $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

   (*v*) Any plane parallel to the given plane
   $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.
   Distance between $ax + by + cz + d_1 = 0$ and
   $ax + by + cz + d_2 = 0$ is $= \dfrac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$.

   (*vi*) **Equation of a plane passing through a given point and parallel to the given vectors:** $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.
   or $\vec{r}\,.(\vec{b}\times\vec{c}) = \vec{a}\,.(\vec{b}\times\vec{c})$ (non parametric form)

9. **A Plane and a Point**

   (*i*) Distance of the point (x', y', z') from the plane $ax + by + cz + d = 0$ is given by $\dfrac{ax' + by' + cz' + d}{\sqrt{a^2 + b^2 + c^2}}$.

   (*ii*) Length of the perpendicular from a point $(\vec{a})$ to plane $\vec{r}.\vec{n} = d$ is given by $p = \dfrac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}$.

   (*iii*) Foot (x', y', z') of perpendicular drawn from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is given by $\dfrac{x' - x_1}{a} = \dfrac{y' - y_1}{b} = \dfrac{z' - z_1}{c} = -\dfrac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$.

(*iv*) **To find image of a point w.r.t. a plane:** Let $P(x_1, y_1, z_1)$ is a given point and $ax + by + cz + d = 0$ is given plane. Let (x', y', z') is the image point then

$$\frac{x'-x_1}{a} = \frac{y'-y_1}{b} = \frac{z'-z_1}{c} = -2\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}.$$

10. **Angle between Two Planes:** $\cos\theta = \left|\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}\right|$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

The angle θ between the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by, $\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}$.

Planes are perpendicular if $\vec{n}_1.\vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$ (λ is a non zero scalar.)

11. **Angle Bisectors**

(*i*) The equations of the planes bisecting the angle between two given planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}.$$

(*ii*) Bisector of acute/obtuse angle: First make both the constant terms positive. Then

$a_1a_2 + b_1b_2 + c_1c_2 > 0 \Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 < 0 \Rightarrow$ origin lies in acute angle

12. **Family of Planes**

(*i*) Any plane through the intersection of

$a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is

$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$

(*ii*) The equation of plane passing through the intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ & $\vec{r}\cdot\vec{n}_2 = d_2$ is $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

13. **Area of Triangle :** From two vector $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then area is given by $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$.

14. **Volume of a Tetrahedron:** Volume of a tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ is given by $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$.

A Line

1. **Equation of a Line**

(*i*) A straight line is intersection of two planes.
It is represented by two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

(*ii*) **Symmetric form:** $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r.$

(*iii*) **Vector equation:** $\vec{r} = \vec{a} + \lambda\vec{b}$.

(*iv*) Reduction of cartesion form of equation of a line to vector form and vice versa

$$\Rightarrow \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \Leftrightarrow \vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$$

2. **Angle between a Plane and a Line**

(*i*) If θ is the angle between line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$, Then

$$\sin\theta = \left[\frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{l^2+m^2+n^2}}\right].$$

(*ii*) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a}+\lambda\vec{b}) = \vec{r}.\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right]$.

(*iii*) Condition for perpendicularity $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$, $\vec{b}\times\vec{n} = 0$.

(*iv*) Condition for parallel $al + bm + cn = 0$, $\vec{b}.\vec{n} = 0$.

3. **Condition for a Line to Lie in a Plane**

(*i*) **Cartesian form:** $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ would lie in a plane

$ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $al + bm + cn = 0$.

(*ii*) **Vector form:** $\vec{r} = \vec{a} + \lambda\vec{b}$ would line in the plane $\vec{r}.\vec{n} = d$ if $\vec{b}.\vec{n} = 0$ & $\vec{a}.\vec{n} = d$.

4. **Skew Lines**

(*i*) The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines. If $\begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} \neq 0$. Then lines are skew.

(*ii*) **Vector Form:** For lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ to be skew $(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1) \neq 0$.

(*iii*) Shortest distance between line $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is $d = \left|\frac{(\vec{a}_2-\vec{a}_1)\times\vec{b}}{|\vec{b}|}\right|$.

5. **Sphere:** General equation of a sphere is
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.
$(-u, -v, -w)$ is the centre and $\sqrt{u^2+v^2+w^2-d}$ is the radius of the sphere.

6. Volume of tetrahedron = 1/3 × height × Area of base

$= \frac{1}{6}$ [Area of parallelepiped]